A FINAL – WED. JUNE 10
8:30 – 10:20 MGH 231

C FINAL – MON. JUNE 8
8:30 – 10:20 SIG 225

MATH 307

HW 7 DUE FRIDAY

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \underbrace{\mathcal{L}\{f(t)\}}_{F(s)}$$

ENTRY TASK Find

REPLACE "t" WITH "t+2"

① $\mathcal{L}\{8 + u_2(t)(t-8)\} = \frac{8}{s} + e^{-2s}\mathcal{L}\{t-6\}$
$= \frac{7}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{6}{s}\right)$

② $\mathcal{L}^{-1}\left\{e^{-4s}\left(\frac{1}{s} + \frac{1}{(s-1)^2} + \frac{s}{s^2+9}\right)\right\} =$

$u_4(t)\, \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{(s-1)^2} + \frac{s}{s^2+9}\right\}(t-4)$

$1 + te^{t} + \cos(3t)$

$u_4(t)\left[(1) + (t-4)e^{(t-4)} + \cos(3(t-4))\right]$

Step Function Laplace Summary

TO COMPUTE $\mathcal{L}\{u_c(t) g(t)\}$

① "Pull out" $u_c(t)$ and make it e^{-cs}
② Replace "t" with "t+c" $\quad e^{-cs}\mathcal{L}\{g(t+c)\}$
③ Use table

TO COMPUTE $\mathcal{L}^{-1}\{e^{-cs} F(s)\}$
① "Pull out" e^{-cs} and make it $u_c(t)$.
② Find $\mathcal{L}^{-1}\{F(s)\}$ in table.
③ Replace "t" with "t−c".

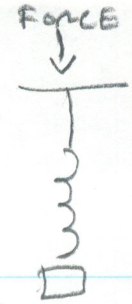
$$f(t) = \begin{cases} \sin(t), & t < 2; \\ t & 2 \le t < 7; \\ 3 & t \ge 7. \end{cases}$$

ASIDE: $8 + u_2(t)(t-8) = \begin{cases} 8, & 0 \le t < 2; \\ t, & t \ge 2 \end{cases}$

JUMP OCCURS HERE

Try it

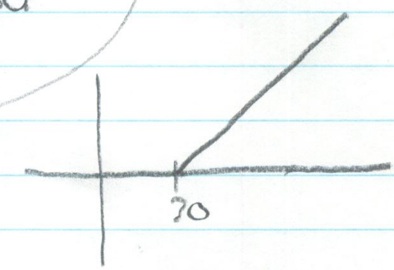
HW QUESTIONS?



Full example

Consider

$$\underbrace{y'' + 4y}_{\text{undamped spring/circuit}} = \underbrace{\begin{cases} 0, & 0 \le t < 30; \\ t-30 & t \ge 30 \end{cases}}_{\text{forcing}}$$



$$y(0)=0, \quad y'(0)=10$$

STEP 1 $g(t) = u_{30}(t)(t-30)$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{u_{30}(t)(t-30)\}$$

STEP 2 $s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = e^{-30s}\mathcal{L}\{t\} = e^{-30s}\frac{1}{s^2}$

$$(s^2+4)\mathcal{L}\{y\} - 10 = e^{-30s}\frac{1}{s^2}$$

$$\mathcal{L}\{y\} = e^{-30s}\frac{1}{s^2(s^2+4)} + \frac{10}{s^2+4}$$

STEP 3 $\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$

$$1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2$$

$s=0 \Rightarrow B = 1/4$

$$1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2$$

$$1 = \underbrace{(A+C)}_{0}s^3 + \underbrace{(B+D)}_{0}s^2 + \underbrace{4As}_{0} + \underbrace{4B}_{1}$$

$A=0, \quad A+C=0 \Rightarrow C=0$

$B=1/4 \quad B+D=0 \Rightarrow D=-B=-1/4$

$$\mathcal{L}\{y\} = e^{-30s}\left(\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right) + \frac{10}{s^2+4}$$

Step 4

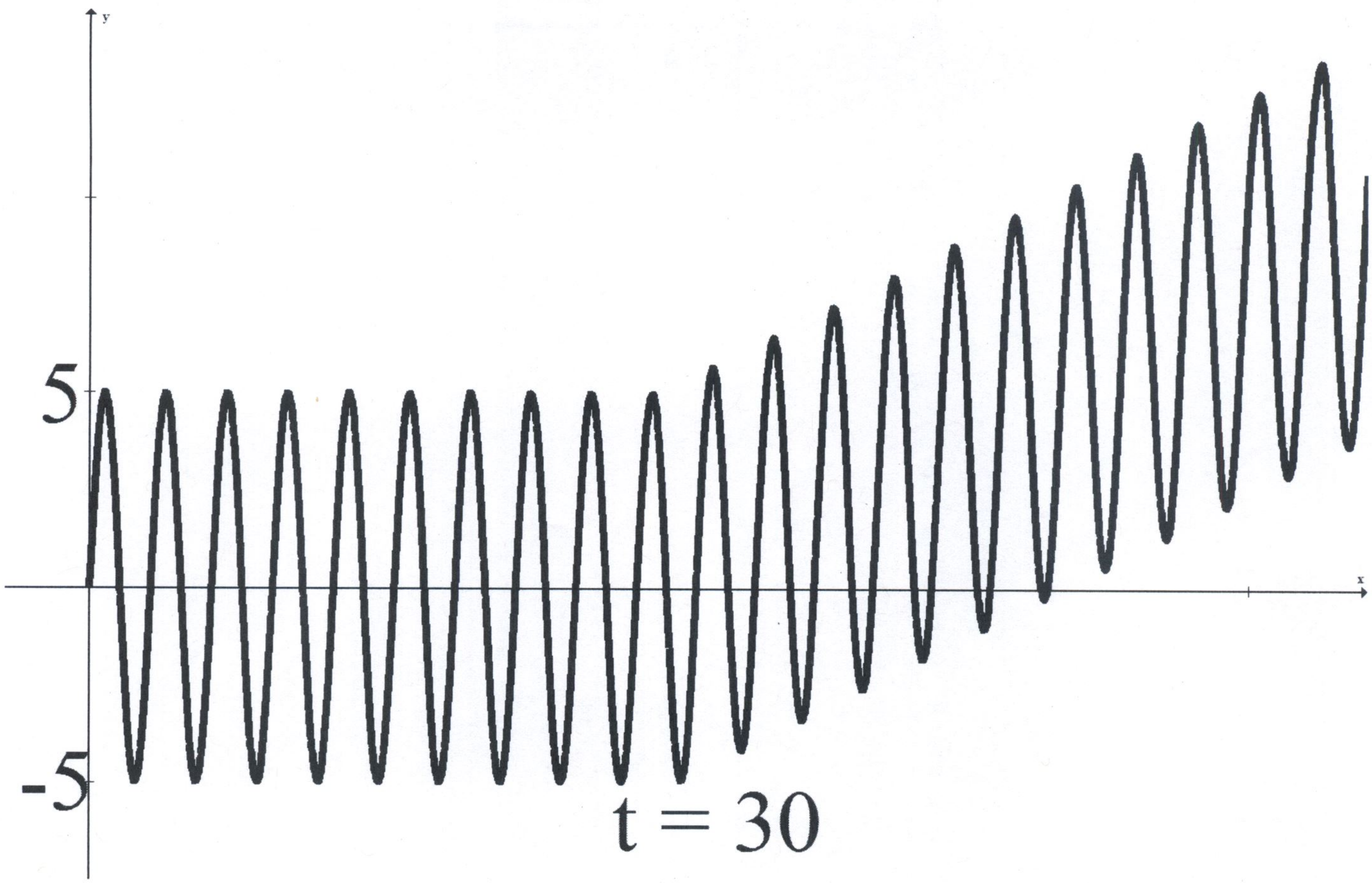
$$y = \mathcal{L}^{-1}\left\{e^{-30s}\left(\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{10}{s^2+4}\right\}$$

$$y(t) = u_{30}(t)\,\mathcal{L}^{-1}\left\{\frac{1/4}{s^2} - \frac{1/4}{s^2+4}\right\}(t-30) + 10\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$\tfrac{1}{4}t - \tfrac{1}{4}\,\tfrac{1}{2}\sin(2t) \qquad\qquad \tfrac{1}{2}\sin(2t)$$

$$\boxed{y(t) = 5\sin(2t) + \tfrac{1}{4}u_{30}(t)\left((t-30) - \tfrac{1}{2}\sin(2(t-30))\right)}$$

$$y(t) = \begin{cases} 5\sin(2t) & , 0 \le t \le 30; \\ 5\sin(2t) + \tfrac{1}{4}(t-30) - \tfrac{1}{8}\sin(2t-60), & t \ge 30 \end{cases}$$


y
5
-5
x
t = 30

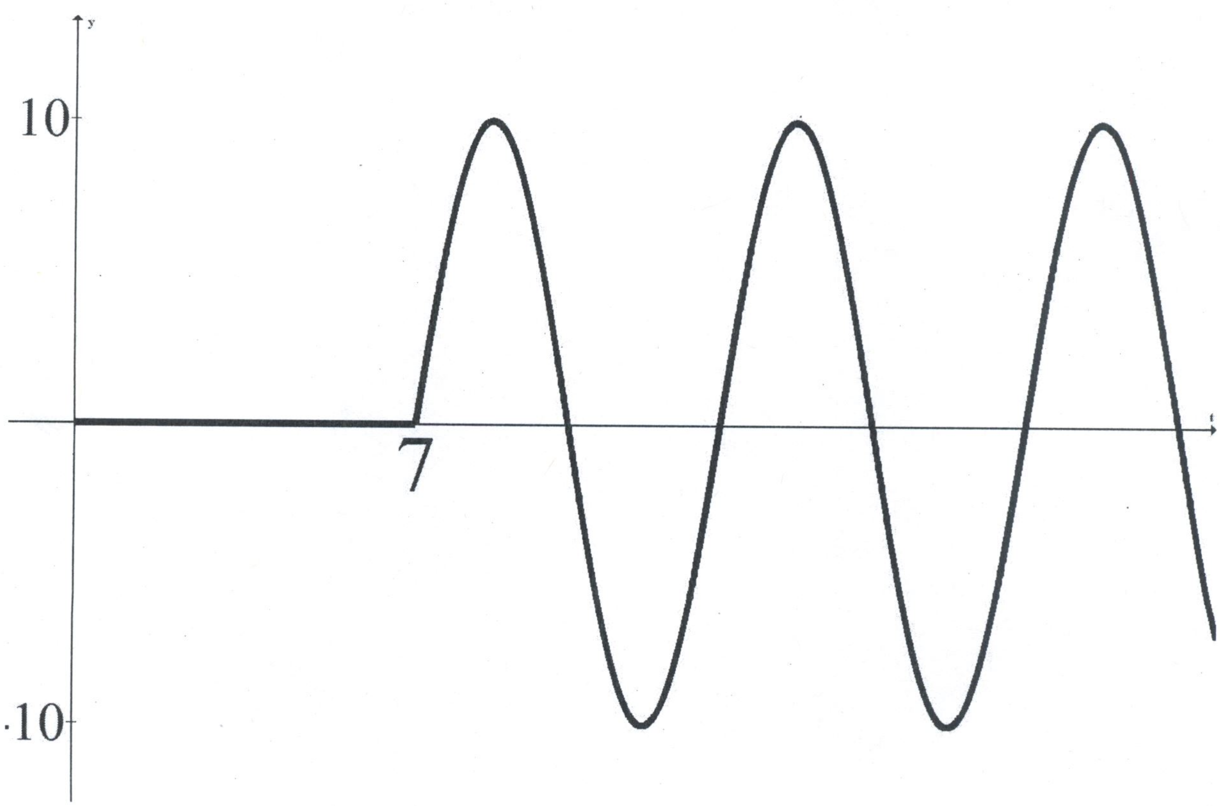
A Full Example of Discontinuous Forcing

Solve

$$y'' + 2y' + y = \begin{cases} 0 & ,0 \leq t < 7; \\ 10\sin(t-7) & ,t \geq 7. \end{cases}$$

with $y(0) = 0, y'(0) = 10$

Here is what the forcing function looks like



1. Rewrite forcing in terms of step functions:

$$f(t) = 10u_7(t)\sin(t-7)$$

2. Laplace transform both sides:

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 10\mathcal{L}\{u_7(t)\sin(t-7)\}$$

3. Using Laplace rules, simplifying, and partial fractions we get

$$(s+1)^2\mathcal{L}\{y\} - 10 = \frac{10e^{-7s}}{s^2+1}$$

$$\mathcal{L}\{y\} = \frac{10e^{-7s}}{(s^2+1)(s+1)^2} + \frac{10}{(s+1)^2}$$

$$= 5e^{-7s}\left(\frac{-s}{s^2+1} + \frac{1}{s+1} + \frac{1}{(s+1)^2}\right) + \frac{10}{(s+1)^2}$$

4. The inverse Laplace transform gives

$$5u_7(t)\left(-\cos(t-7) + e^{-(t-7)} + (t-7)e^{-(t-7)}\right) + 10te^{-t}$$

Here is a graph of the solution: